Math 3280A 22-10-06

Review

- Cumulative distributions

$$F_X(b) = P\{X \le b\}, \quad b \in \mathbb{R}.$$

F_X is non-decreasing, right cts, $\lim_{b\to+\infty} F_X(b) = 1$, $\lim_{b\to-\infty} F_X(b) = 0$

$$P\{X = b\} = F_X(b) - F_X(b-).$$

- Continuous r.v.'s.

X is said to be (absolutely) continuous if $\exists$ a nonnegative function on $\mathbb{R}$ such that

$$P\{X \in B\} = \int_B f(x)\,dx$$

for any "measurable" set $B \subset \mathbb{R}$.

The function f is called the probability density function of X, or simply pdf of X.

<u>Remark</u>: In the continuous case.

$$P\{X = a\} = 0 \quad \text{for any } a \in \mathbb{R}.$$

Hence $P\{a \le X \le b\} = P\{a < X \le b\} = P\{a \le X < b\} = P\{a < X < b\}$.

Exer. 1. Suppose X has a pdf

$$f(x) = \begin{cases} \lambda e^{-\frac{x}{100}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(1) Find the value of λ.

(2) Find $P\{X > 100\}$.

Solution:

$$1 = \int_{-\infty}^{\infty} f(x)\, dx$$

$$= \int_0^{\infty} \lambda e^{-\frac{x}{100}}\, dx$$

$$= \lambda \cdot \left(-100\, e^{-\frac{x}{100}}\right)\Big|_0^{\infty}$$

$$= \lambda \cdot 100$$

Hence $\lambda = \frac{1}{100}$.

$$P\{X > 100\} = \int_{100}^{+\infty} f(x)\,dx$$

$$= \int_{100}^{+\infty} \frac{1}{100} e^{-\frac{x}{100}}\,dx$$

$$= -e^{-\frac{x}{100}} \Big|_{100}^{\infty}$$

$$= e^{-1}. \qquad \boxed{//}$$

§ 5.2 Expectation of a cts. r.v.

Recall that the expectation of a discrete r.v. X is defined by

$$E[X] = \sum x\,P\{X = x\},$$

where the summation is taken over all the possible values that X can take on.

We can not directly use this method to define the expectation of a cts r.v., since in the cts case, X will take on uncountably diff values, and moreover,

$$P\{X = x\} = 0 \text{ for all } x \in \mathbb{R}.$$

Def. Let X be a cts r.v. with pdf f.

Then we define

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Intuitive idea:

In the continuous case, we make a partition of $(-\infty, \infty)$ by $(x_n)_{n=-\infty}^{\infty}$ such that

$$x_{n+1} - x_n = \Delta x.$$

Then

$$\sum_n x_n \cdot P\{x_n < X \leq x_{n+1}\}$$

$$= \sum_n x_n \int_{x_n}^{x_n+\Delta x} f(x)\, dx$$

$$\approx \sum_n x_n f(x_n) \cdot \Delta x \longrightarrow \int_{-\infty}^{\infty} x f(x)\, dx$$

as $\Delta x \to 0$.

Example 2. X is said to be uniformly distributed on $[0,1]$ if it has the following density

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Find $E[X]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x \cdot 1\, dx$$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}. \quad \square$$

Below we consider the expectation of functions of cts r.v.'s.

Prop 3. Let X be a cts r.v. with density f.

Let $g: \mathbb{R} \to \mathbb{R}$. Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

We prove the above prop only in the case that $g \geq 0$.

To this end, we first prove the following.

Lem 4. Let Y be a non-negative cts r.v. Then

$$E[Y] = \int_0^{\infty} P\{Y > y\}\, dy.$$

To prove Lem 4, we need to use the following result:

$$\int_a^b \left(\int_c^d f(x,y)\, dx \right) dy = \int_c^d \left(\int_a^b f(x,y)\, dy \right) dx$$

when f is non-negative. This is a special version of the Fubini Theorem.

Pf of Lem 4. Let f be the density of Y.

Since Y is non-negative, $f(x)=0$ for $x \leq 0$.

So $E[Y] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x f(x)\,dx$. Observe

$$\int_0^{\infty} P\{Y > y\}\,dy$$

$$= \int_0^{\infty} \left(\int_y^{\infty} f(x)\,dx \right) dy$$

$$= \int_0^{\infty} \left(\int_y^{\infty} 1 \cdot f(x)\,dx + \int_0^{y} 0 \cdot f(x)\,dx \right) dy$$

$$= \int_0^{\infty} \left(\int_0^{\infty} g(x,y) f(x)\,dx \right) dy,$$

(where $g : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is defined by

$$g(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y. \end{cases}$$
)

$$\overset{\text{Fubini}}{=} \int_0^{\infty} \left(\int_0^{\infty} g(x,y) f(x)\,dy \right) dx$$

$$= \int_0^{\infty} f(x) \left(\int_0^{\infty} g(x,y)\,dy \right) dx$$

$$= \int_0^\infty f(x) \left(\int_0^x 1 \, dy + \int_x^\infty 0 \, dy \right) dx$$

$$= \int_0^\infty x f(x) \, dx$$

$$= E[Y]. \qquad \blacksquare$$

Proof of Prop. 3: By Lem 4,

$$E[g(X)] = \int_0^\infty P\{ g(X) > y \} \, dy.$$

Write $B := \{ x \in \mathbb{R} : g(x) > y \}$. Then

$$P\{ g(X) > y \} = P\{ X \in B \} = \int_B f(x) \, dx = \int_{\{x : g(x) > y\}} f(x) \, dx.$$

Hence

$$E[g(X)] = \int_0^\infty \int_{\{x : g(x) > y\}} f(x) \, dx \; dy$$

$$= \int_0^\infty \left(\int_{-\infty}^\infty h(x,y) \; f(x) \, dx \right) dy$$

where $h(x,y) = \begin{cases} 1 & \text{if } g(x) > y \\ 0 & \text{otherwise}. \end{cases}$

So by the Fubini Thm,

$$E[g(X)] = \int_{-\infty}^{\infty}\left(\int_{0}^{\infty} h(x,y)\, f(x)\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_{0}^{\infty} h(x,y)\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_{0}^{g(x)} h(x,y)\, dy + \int_{g(x)}^{\infty} h(x,y)\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_{0}^{g(x)} 1\, dy + \int_{g(x)}^{\infty} 0\, dy\right) dx$$

$$= \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

$\square$.